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# Heterotopic clustering

Bernard J. T. Jones,[1]★ Peter Coles[2] and Vicent J. Martínez[3]
[1] *Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QH*
[2] *Astronomy Unit, School of Mathematical Sciences, Queen Mary & Westfield College, Mile End Road, London E1 4NS*
[3] *Departament de Matemàtica Aplicada i Astronomia, Universitat de València, Burjassot, 46100 València, Spain*

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## SUMMARY

The observed galaxy clustering appears to display a rich scaling structure that can be characterized as a multifractal. We present two mathematical models of the non-linear clustering regime based on these interwoven fractal point sets, and show how the multifractal description provides some important insights into what is going on in non-linear clustering. This approach contrasts with earlier hierarchical models for non-linear galaxy clustering in which each level of the hierarchy is a rescaled version of its ancestor level; such hierarchies yield simple fractal distributions characterized by a single scaling exponent,

The possibility of using a multifractal model to characterize the scaling properties of the clustering has important consequences. We show that being a multifractal is related to the galaxy counts-in-cells following a skewed statistical distribution such as the Lognormal. The model predicts the scaling properties of the mean and variance of the cell-count distribution with cell size. Heterotopic point distributions are a manifestation of point sets that are Lognormally distributed; in these models the voids are associated with a kind of *spatial* intermittency ('heterotopicity').

A multifractal model also provides insights into the way the non-linear clustering process works. We present a non-linear fragmentation model that is manifestly hierarchical in nature and that reproduces the complex multifractal scaling behaviour observed in galaxy clustering in the non-linear regime. Unlike the simple scaling hierarchies which describe non-linear clustering in terms of a single scaling parameter, this model has four scaling parameters that describe the statistical redistribution of material at each level of the clustering hierarchy.

**Key words:** galaxies: clustering – galaxies: formation – large-scale structure of Universe.

## 1 INTRODUCTION

Since the early 1970s, galaxy clustering has been described mathematically in terms of two-point and higher order correlation functions. The two-point function has, of course, been important because of its close link with the way in which the gravitational potential drives the clustering process in the non-linear regime via the so-called cosmic virial theorem (Peebles 1980). Higher order correlation functions have not added substantially to the description of the clustering, but the underlying scaling relations expressing the higher order functions as sums of products of two-point functions have strengthened the impression that we are seeing the results of a gravitation-driven clustering hierarchy (Peebles 1980; Schaeffer & Silk 1985; Balian & Schaeffer 1989a,b). The existence of these scaling relationships, and the fact that in the non-linear regime the two-point correlation function is well approximated by a simple power law, suggested to Efstathiou, Fall & Hogan (1979) that the clustering may be described as a simple fractal hierarchy. Such a hierarchy is characterized by one exponent (scaling index): the slope of the two-point correlation function. All higher orders of correlation functions are directly related to sums of products of two-point functions.

Analysis of three-dimensional galaxy catalogues shows, however, that the distribution is not a simple homogeneous fractal (Jones *et al.* 1988; Martínez & Jones 1990), though it

★Present address: Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark.

does nevertheless have scaling properties. The distribution can be described as a kind of inhomogeneous fractal called a *multifractal* (see, for example, Halsey *et al.* 1986; Martínez *et al.* 1990; Martínez 1991).

The description of the scaling properties of the observed galaxy distribution in terms of multifractal scaling is both mathematically elegant and statistically powerful. However, it remains somewhat unclear just what such multifractal scaling means in terms of the perceived spatial pattern of the galaxy distribution or in terms of the dynamics of non-linear clustering. It is mainly the first of these issues that we shall tackle in this paper, though our model of multiscale hierarchical clustering does throw some light on the dynamical question.

We shall demonstrate that the multifractal nature of spatial fluctuations implies what would, for a stochastic process in the time domain, be described as 'intermittency' but in the spatial domain is more correctly termed 'patchiness': the distribution consists of large expanses devoid of structure separated by high-density regions. We shall use the term *heterotopic* to describe a patchy spatial distribution of this sort.† We use this word to avoid the implication that the dynamical origin of this patchiness is the same as the origin of intermittency in fully developed turbulence. We shall also demonstrate that heterotopicity is associated with the density having a skewed statistical distribution such as that typified by the Lognormal distribution (Coles & Barrow 1987; Coles & Jones 1991). This is again analogous to the intermittent behaviour observed in time-domain stochastic processes where long quiescent intervals are interrupted by sporadic, highly active bursts (Kolmogorov 1962; Zel'dovich *et al.* 1985, 1987; Paladin & Vulpiani 1987). Castagnoli & Provenzale (1990) have already built a fractal cascade model for galaxy clustering based on a generalization of the random $\beta$-model for fully developed turbulence.

The plan of this paper is first to review the description of a multifractal in terms of its singularity spectrum, 'the $f(\alpha)$ spectrum', and to cast this into a form needed for this problem. We then go on to derive the Lognormal model and the correlation function and discuss the meaning of these results and some wider questions surrounding the multifractal formalism.

It should be emphasized that the attention of this paper is focused on the *non-linear clustering* regime where we have every reason to expect scaling properties. It has been customary to describe the galaxy distribution on the sky as 'filamentary' (Peebles 1980 and references therein). The first substantial redshift survey, the 'CfA Slice' (de Lapparent, Geller & Huchra 1986, 1988, 1991), showed that the galaxy distribution on large scales in fact comprises a network of filaments and sheets surrounding a general distribution of voids, and subsequent surveys have strengthened that impression (Geller & Huchra 1989; Broadhurst *et al.* 1990; Babul & Postman 1990; Saunders *et al.* 1991). Most of that structure is seen on scales in excess of 20 $h^{-1}$ Mpc and so falls outside the scope of this paper. Our use of the word 'voids' in this paper should not evoke pictures of these large-scale features.

†The word is derived from the greek *hetero-* meaning 'different' and *topos* meaning, literally, 'terrain'.

## 2 A MULTIFRACTAL MODEL

The multifractal model is extensively reviewed from a mathematical point of view in the article of Martínez *et al.* (1990). The strict definitions of the various dimensions used to describe point sets are discussed in that paper and shall not concern us here. In this paper, we shall permit ourselves the liberty of ignoring the subtle (but none the less important) differences between, say, Renyi dimensions, box-counting dimensions, Kolmogorov capacity and the rest. In most of what follows we shall in fact use the Grassberger & Procaccia (1983) definitions of dimension, but we shall try to keep our arguments as intuitive as possible.

### 2.1 $f(\alpha)$

Let us suppose that the galaxy distribution tends to avoid certain regions of space which we can refer to as 'voids'. In the non-void regions, let the mass of a sphere of radius $r$ centred at an arbitrarily chosen point $\boldsymbol{x}$ scale as

$$m(r, \boldsymbol{x}) \propto r^{\alpha(\boldsymbol{x})}. \tag{1}$$

The density averaged over such a sphere therefore varies as

$$\rho(r, \boldsymbol{x}) \propto r^{\alpha(\boldsymbol{x})-3}. \tag{2}$$

The $\alpha(\boldsymbol{x})$ thus define a set of scaling indices or *pointwise dimensions* in terms of the local scaling properties of the mass distribution:

$$\alpha(\boldsymbol{x}) = \lim_{r \to 0} \frac{\log m(r, \boldsymbol{x})}{\log r}. \tag{3}$$

If the galaxy distribution were a homogeneous fractal then $\alpha$ would be constant irrespective of position, but for a multifractal the index $\alpha$ varies from neighbourhood to neighbourhood in an arbitrary fashion. We shall make the further assumption that the set of points such that $\alpha$ is constant itself forms a fractal of dimension $f(\alpha)$ (see Halsey *et al.* 1986). This is only a first approximation to the full physical meaning of $f(\alpha)$ and is only true for some particular kinds of multifractal sets (Grassberger, Badii & Politi 1988; Falconer 1990). Since $\alpha$ varies over the occupied volume, the model consists of interwoven fractals on which the scaling index takes different values. We refer to such a distribution as a multifractal for obvious reasons.

Being a multifractal is rather special and implies two things. First, a given value of $\alpha$ corresponds to one particular dimension $f(\alpha)$. This is not unreasonable if one is to argue it in favour of global homogeneity, though it is not actually demanded by homogeneity. Secondly, the $f(\alpha)$ curve has only one maximum. Other fractal constructs are conceivable; one such is the Balian–Schaeffer (1989a, b) bifractal in which two homogeneous fractals having different dimensions are interwoven. It is the fact of being a multifractal that will allow us to approximate the function $f(\alpha)$ as a quadratic in the neighbourhood of its unique maximum, and that enables us to draw far-reaching conclusions about the counts-in-cells distribution.

Of course, when dealing with a discrete point set one cannot take the limit expressed in (3) in a meaningful way. It is possible, however, to work with the $\alpha$ evaluated at some finite resolution $r$ through the definition of a set of *crowding indices*:

$$\hat{\alpha}(\boldsymbol{x})=\frac{\log m(r,\boldsymbol{x})}{\log r} \tag{4}$$

(Grassberger *et al.* 1988), where $r$ is a small but finite resolution scale which must be greater than or of order the mean distance between points.

Multifractal distributions of galaxies in the Universe can be characterized in terms of the so-called $f(\alpha)$ distribution which contains information about the dimensionality of the set of iso-$\alpha$ points, i.e. those points where the scaling index is the same. The $f(\alpha)$ curve has a maximum at some value $\alpha_0$ where $f(\alpha)=D_0$, the box-counting dimension of the set. This is probably the easiest fractal dimension to understand at the intuitive level: if the volume $V$ is covered by cells of side $r$ (a dimensionless quantity, expressed in units of $V^{1/3}$), then the number of occupied cells will scale as

$$N(r)\propto r^{-D_0} \tag{5}$$

so that

$$D_0=\lim_{r\to 0}\frac{\log N(r)}{\log(1/r)}\simeq\frac{d\log N(r)}{d\log(1/r)}. \tag{6}$$

To be pedantic, the equation (6) is actually the definition of the Kolmogorov capacity of the set, which is an upper bound on its Hausdorff dimension. We can generalize equation (5) by considering the number of boxes needed to cover the subset characterized by a given $\alpha$ value:

$$N_\alpha(r)\propto r^{-f(\alpha)}. \tag{7}$$

### 2.2 $D_q$

The Kolmogorov capacity does not provide us with a full description of all the scaling properties of a date set. A more complete description of the multifractal behaviour is obtained via its set of generalized dimensions $D_q$. If the sample volume in which particles (galaxies) are distributed is divided into $N$ cells of linear size $r$ and the $i$th cell then has $n_i(r)$ particles, then

$$D_q=\lim_{r\to 0}\frac{1}{(q-1)}\frac{\log\sum_{i=1}^{N(r)}[n_i(r)/N]^q}{\log r} \tag{8}$$

(for $q\neq 1$). $D_q$ describes how the $q$th moment of the number count distribution scales with cell size, and thus encapsulates the scaling properties of the moments of the cell counts in one convenient function. There is no information about the relative amplitudes of the different moments. Again, the limit in equation (8) cannot be taken for finite point sets, but if the moments satisfy the scaling relationship

$$\sum_{i=1}^{N(r)}\left[\frac{n_i(r)}{N}\right]^q\propto r^{(q-1)D_q} \tag{9}$$

over a certain scaling range, we can obviously estimate $D_q$. There must be a significant range of $r$ over which $D_q$ is defined in order to conclude that the point distribution has the scaling properties associated with multifractal behaviour. Under these circumstances there is (perhaps surprisingly) a relationship between the functions $D_q$ and $f(\alpha)$: they are related through a Legendre transform (Halsey *et al.* 1986; Martínez *et al.* 1990).

Although we shall work mostly in terms of $f(\alpha)$, we shall need the relationship between the $D_q$ and $f(\alpha)$. First define a function $\tau(q)$ by the Legendre transform:

$$\tau(q)=\alpha q-f(\alpha),\qquad \alpha(q)=\frac{d\tau}{dq}. \tag{10}$$

The generalized dimension, $D_q$, is then

$$D_q=(q-1)^{-1}\tau(q). \tag{11}$$

The relationship between $f(\alpha)$ and $\tau(q)$ is illustrated in Fig. 1. As discussed extensively elsewhere, the $D_q$ are related to the scaling properties of the moments of the distribution of the density. It is important to notice that, despite the implication of its name, $D_q$ does not represent a *dimension* for every value of $q$. In particular, it may happen that $D_q>d$ (the Euclidean dimension of the set within which it is embedded) for $q<0$.

From these equations it is immediately apparent that a homogeneous fractal (where $\alpha$ is constant) gives a degenerate $f(\alpha)$ spectrum (given by a single point) and also has all generalized dimensions $D_q$ equal to $D_0$.

## 3 STATISTICS OF GALAXY CLUSTERING IN THE MULTIFRACTAL MODEL

### 3.1 Cell count statistics

We now wish to relate the multifractal scaling properties of this simple model to more orthodox statistical descriptors. In particular, we shall attempt to derive the distribution of cell-counts and the low-order moments of this distribution. A similar analysis has, in fact, already been used to study the dissipation field in turbulent flows (Meneveau & Sreenivasan 1987). For clarity, and because of the somewhat different notation, we shall repeat the main points of the derivation here. Assuming that the density field around a point obeys the local scaling law (1), we can relate the crowding index to the density field smoothed on some characteristic scale $r_s$. Typically, $r_s$ is the linear size associated with the sample volume, i.e. $r_s=V^{1/3}$. Denoting the cell occupancy by $\eta_r$, we have

$$\alpha=\frac{\log(\eta_r/N)}{\log(r/r_s)} \tag{12}$$

(dropping the accent on the $\alpha$ since we shall be working with the point-set definitions from here on). $N$ is the total number of points in the set. The probability that the occupancy around a given point *not in a void* is $\eta_r$ is then given by

$$P(\eta_r)=\frac{1}{P(\eta_r\neq 0)}P(\alpha)\frac{d\alpha}{d\eta_r}. \tag{13}$$

The factor $P(\eta_r\neq 0)$ takes account of the fact that galaxies are excluded from some volumes, the voids, where the density of galaxies is zero. This avoids unphysical negative

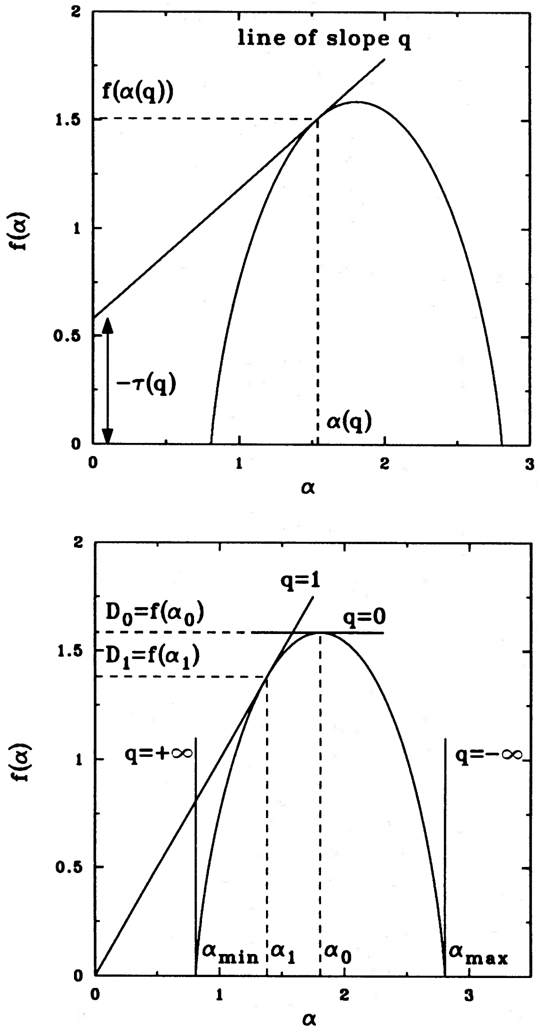


**Figure 1.** The scaling exponents $\tau(q)$ and the multifractal spectrum are related by a Legendre transformation, as can be seen in the upper plot. Some important values of the $f(\alpha)$ curve are shown in the lower plot, along with their corresponding $D_q$ values.

densities in the final distribution. Now the Kolmogorov capacity is determined by the fraction of occupied cells (i.e. places where $\eta_r \neq 0$) through (5) and therefore scales as

$$P(\eta_r \neq 0) \propto \left(\frac{r}{r_s}\right)^{3-D_0}. \tag{14}$$

Similarly, since $\alpha$ has a constant value on a fractal of dimension $f(\alpha)$ – see equation (7) – we obtain

$$P(\alpha) \propto \left(\frac{r}{r_s}\right)^{3-f(\alpha)}. \tag{15}$$

Collecting up powers of $r$, we obtain the scaling relationship

$$P(\eta_r) \propto \frac{N}{\eta_r} \frac{1}{\log(r/r_s)} \left(\frac{r}{r_s}\right)^{D_0-f(\alpha)}, \tag{16}$$

where the last term here is just the derivative $d\alpha/d\eta_r$ in (13).

### 3.2 A parabolic approximation to $f(\alpha)$

To complete our argument, we now need to remove references to $f(\alpha)$ and write the distribution in terms of $\eta_r$ and $r$. This is done by noting that the data suggest that one might make a parabolic approximation to the function $f(\alpha)$ over a considerable range of $\alpha$ (Martínez *et al.* 1990).

It was first remarked by Meneveau & Sreenivasan (1987) that $f(\alpha)$ for a turbulent dissipation field could be approximated as a parabola in the vicinity of the maximum of $f(\alpha)$. From equations (10) we see that $f'(\alpha)=q$, so that the maximum of $f(\alpha)$ occurs at the value $\alpha=\alpha_0$ corresponding to $q=0$. Meneveau & Sreenivasan also remarked that any parabolic function used to approximate $f(\alpha)$ is constrained by the requirements arising out of equations (10) that at $q=1$, $\alpha(1)=f[\alpha(1)]$ and $f'[\alpha(1)]=1$. The quadratic in $\alpha$ that does the job is

$$f(\alpha)=D_0-\frac{(\alpha-\alpha_0)^2}{4(\alpha_0-D_0)}. \tag{17}$$

The curvature at the maximum is $f''(\alpha_0)=-1/2(\alpha_0-D_0)$. [See also Paladin & Vulpiani (1987) for another discussion of this in relation to turbulence.]

With this approximation, we find that (16) becomes

$$P(\eta_r) \propto \frac{N}{\eta_r} \frac{1}{\log(r/r_s)} \left(\frac{r}{r_s}\right)^{[(\alpha-\alpha_0)^2/4(\alpha_0-D_0)]}. \tag{18}$$

Now we can substitute for $\alpha$ in terms of $\eta_r$ from (12) and rewrite the power of $r$ as an exponential. The result is

$$P(\eta_r) \propto \left(\frac{N}{\eta_r}\right) \frac{1}{\log(r/r_s)} \exp\left\{-\frac{[\log(\eta_r/N)-\alpha_0 \log(r/r_s)]^2}{4(\alpha_0-D_0)\log(r_s/r)}\right\}, \tag{19}$$

which is instantly recognisable as a Lognormal distribution; in other words the variable $\log \eta_r$ is normally distributed with mean and variance given by

$$\mu=\log N-\alpha_0 \log\frac{r_s}{r}, \qquad \sigma^2=2(\alpha_0-D_0)\log\frac{r_s}{r}, \tag{20}$$

respectively. Following the usage in the theory of turbulence we can call the quantity $2(\alpha_0-D_0)$ the *Lognormal Intermittency Exponent* on the scale $r$ (Kolmogorov 1962; Zel'dovich *et al.* 1985, 1987; Paladin & Vulpiani 1987).

### 3.3 Putting in some numbers

It is interesting to demonstrate the relevance of this approximation to empirically determined properties of galaxy clustering. From the Martínez *et al.* (1990) analysis of the CfA data, we have

$$\alpha_0=2.6, \qquad D_0=2.1\pm0.1. \tag{21}$$

We now use the expression (17) for the quadratic approximation to the $f(\alpha)$ spectrum. From the Legendre transform relations (10), we know that

$$q=\frac{df(\alpha)}{d\alpha}=\frac{\alpha-\alpha_0}{2(D_0-\alpha_0)}. \tag{22}$$

Inverting this expression leads to

$$\alpha(q)=2(D_0-\alpha_0)\,q+\alpha_0 \tag{23}$$

so that, using the Legendre transform relations (10) again, we get

$$\alpha(q)=\frac{d\tau(q)}{dq}=2(D_0-\alpha_0)\,q+\alpha_0. \tag{24}$$

Simple integration leads to

$$\tau(q)=(D_0-\alpha_0)\,q^2+\alpha_0 q+C, \tag{25}$$

where $C$ is an arbitrary constant. We know, however, that $\tau(0)=-D_0$ so that $C=-D_0$ [one could as an alternative use the fact that $\tau(1)=0$ to derive the same result]. The set of generalized dimensions is therefore

$$D_q=(q-1)^{-1}\tau(q)=(q-1)^{-1}[(D_0-\alpha_0)\,q^2+\alpha_0 q-D_0]$$
$$=(D_0-\alpha_0)\,q+D_0. \tag{26}$$

This shows that $D_q$ for the parabolic $f(\alpha)$ spectrum (17) is simply a straight line (Nelkin 1989). This line is compared with the data in Fig. 1: it gives a remarkable fit over the range $-1<q<2$. Inserting the empirically determined values of $D_0$ and $\alpha_0$ (21) leads to

$$D_2=1.1\pm0.3 \tag{27}$$

for the *correlation dimension* (recall that the correlation dimension and Hausdorff dimension are equal for a purely homogeneous fractal). This shows not only that the galaxy distribution is not described by a homogeneous fractal, but also that our quadratic approximation to $f(\alpha)$ is consistent with the two-point correlation properties of the galaxies. If the two-point galaxy–galaxy correlation function is

$$\xi(r)=\left(\frac{r}{r_0}\right)^{-\gamma}, \tag{28}$$

then the correlation dimension $D_2$, on small scales where $\xi(r)\gg1$, should be given by

$$D_2=3-\gamma. \tag{29}$$

Since $\gamma$ is in the range 1.7 to 1.8 (Davis & Peebles 1983; Shanks *et al.* 1983), this is consistent with our treatment.

It is worth also remarking that the $D_q$ we have derived represents a decreasing function of $q$:

$$\frac{dD_q}{dq}=\frac{(D_0-\alpha_0)(q^2+1)}{(q-1)^2}<0. \tag{30}$$

In fact, this condition is necessary for the description to be consistent and it contrasts in that respect with the bifractal model of Balian & Schaeffer (1989b). Finally, in the same spirit, let us look at the variance of $\eta_r$ obtained using the quadratic approximation (17). Using known properties of the moments of the Lognormal distribution (Coles & Jones 1991), we find that the mean and variance of $\eta_r$ are just

$$\langle\eta_r\rangle=\exp(\mu+\tfrac{1}{2}\sigma^2),$$
$$\Sigma^2(r)=\exp(2\mu+\sigma^2)[\exp(\sigma^2)-1]. \tag{31}$$

If we use the expressions (20) for $\mu$ and $\sigma^2$ in terms of $r$ and $\alpha_0$, we find that

$$1+\frac{\Sigma^2(r)}{\langle\eta_r\rangle^2}\propto r^{-2(\alpha_0-D_0)}\sim r^{-1} \tag{32}$$

for $\alpha_0\simeq2.6$ and $D_0\simeq2.1$. Note that the simple fractal clustering hierarchy leads to this quantity being constant, since the distribution scales in the same way at every point (Peebles 1980). The behaviour (32) is therefore a signal of heterotopicity: if the intermittency exponent $2(\alpha_0-D_0)$ were zero, $D_q=D_0$ for all $q$ and we would recover the simple fractal scaling behaviour when $\xi(r)\gg1$.

Note further that we can deduce the behaviour of the two-point correlation function directly from the correlation dimension, $D_2$:

$$1+\xi(r)\propto r^{3-D_2}\sim r^{-1.7\pm0.1}. \tag{33}$$

This demonstrates the consistency of our approximation both internally and in comparison with the data.

## 4 HIERARCHICAL CLUSTERING

### 4.1 A multiplicative random process

We are now in a position to provide an alternative to the simple hierarchical clustering models of the kind presented by Peebles (1974), Gott & Rees (1975) and Efstathiou *et al.* (1979). These models result in simple fractal distributions and hence produce $f(\alpha)$ curves that are delta functions. The idea is to construct a multifractal process that generates the same $f(\alpha)$ curve as suggested by the data (Martínez *et al.* 1990). The simple quadratic approximation to $f(\alpha)$ presented in the previous sections does not lend itself to an intuitive physical interpretation, except to say that the galaxy distribution is almost Lognormal, and so the $f(\alpha)$ curve is approximated by a particular quadratic function of $\alpha$. However, the multiplicative model introduced by Martínez *et al.* (1990) in the context of galaxy clustering directly reflects the hierarchical nature of non-linear galaxy clustering.

The multiplicative model is based on constructing a nested hierarchy of $2\times2\times2$ cells. (Other similar constructions based on different subdivisions and hierarchy rules are of course possible; see for example the rescaling hierarchy of Hentschel & Procaccia 1983). On the largest scale the sample volume is divided into eight cells, each of which is assigned a fraction $f_i$ of all the matter, such that

$$\sum_{i=1}^{8} f_i=1, \qquad \text{for} \qquad 0\le f_i\le1. \tag{34}$$

Each of these cells is then itself subdivided into eight cells and the matter in the parent cell redistributed according to the same set of numbers $f_i$. The order in which the eight subcells are assigned the values $f_i$ is permuted randomly. The value of the density attached to a cell is thus proportional to the product of the $f_i$s attached to that cell, its parents and all of its ancestors. If any of the $f_i$ in this tree is zero, that cell has zero density, as do all its descendants. It is the fact that some of the $f_i$ are zero that causes the heterotopic nature of the clustering. Note that the largest cell should correspond to the largest non-linear scale in the clustering distribution. We have no prescription for relating cells above this scale and so can say nothing about the clustering structure on the very largest scales.

It was shown by Martínez *et al.* (1990) that the generalized dimensions for this distribution tend to

$$D_q = (1-q)^{-1} \log_2 \sum_{i=1}^{8} f_i^q. \quad (35)$$

The Legendre transform (10) gives $\alpha$ as a function of $q$:

$$\alpha(q) = -\frac{\sum_{i=1}^{8} f_i^q \log_2 f_i}{\sum_{i=1}^{8} f_i^q} \quad (36)$$

and

$$f[\alpha(q)] = \frac{\left(\sum_{i=1}^{8} f_i^q\right)\left(\log_2 \sum_{i=1}^{8} f_i^q\right) - q \sum_{i=1}^{8} f_i^q \log_2 f_i}{\sum_{i=1}^{8} f_i^q}. \quad (37)$$

Using (35), we see immediately that

$$D_0 = \log_2 \sum_{i:f_i \neq 0} 1, \quad (38)$$

and so, if all the $f_i$ are non-zero, $D_0 = 3$, the dimension of the space in which the points are distributed. This describes what is known as a 'fat' fractal. If only four of the $f_i$ are non-zero, $D_0 = 2$, whereas if five are non-zero, $D_0 \simeq 2.32$. We therefore choose the case where four of the $f_i$ are non-zero, since this gives a $D_0$ that is certainly consistent with the value $D_0 = 2.1 \pm 0.1$ deduced from the data.

### 4.2 Getting the parameters from the data

The problem is now to select $f_i$, $i = 1 \ldots 4$ so that the multiplicative model fits the $f(\alpha)$ curve, or equivalently, the $\tau(q)$ curve, inferred from the data. As already remarked, choosing only four non-zero $f_i$ that sum to unity already guarantees that $\tau(0)$ and $\tau(1)$ are correct. One constraint on the $f_i$ is that the maximum of $f(\alpha)$ falls at $\alpha_0 = 2.6$ ($q = 0$). Another constraint is to get the slope of the two-point correlation function; for this we need $\tau(2) = 1.3$, consistent with our choice of $\alpha_0$. We can further constrain the $f_i$ by fitting some other point on the $\tau(q)$ curve, for example $\tau(-1) = -5.1$. Thus we solve the four constraints:

$$\sum_{i=1}^{4} f_i = 1, \qquad -\log_2 \sum_{i=1}^{4} f_i^2 = 1.3,$$

$$-\log_2 \sum_{i=1}^{4} f_i^3 = -5.1, \qquad -\sum_{i=1}^{4} \log_2 f_i = 10.4. \quad (39)$$

Solving the equations (39) yields

$$f_1 = 0.07(\pm 0.01), \qquad f_2 = 0.07(\pm 0.01),$$

$$f_3 = 0.32(\pm 0.02), \qquad f_4 = 0.54(\pm 0.03). \quad (40)$$

We might have chosen different $q$ values at which to constrain the $\tau(q)$ curve, and we would have obtained rather similar $f_i$. The quoted errors are indicative of the error introduced by using a range of fitting values from different constraints. To illustrate the visual appearance of this model, we show in Fig. 2 a 3D box in which a realization has been performed. The width of the slice is 10 per cent of the box side. The multiplicative model has been generated using the parameters given in (40). These values yield a fit to $\tau(q)$ for $-1.5 \leq q \leq 2.5$ that is generally better than a few per cent over the whole range, certainly better than is warranted by the data!

The CfA data, the quadratic approximation (17) and (26) and the multiplicative multifractal approximation to the $D_q$ and $f(\alpha)$ curves are shown in Fig. 3. The significance of these numbers is two-fold. First, since only four of the $f_i$ are non-zero, half of the volume is swept clean at each level of the hierarchy. It is that which yields the correct Hausdorff dimension and gives rise to the heterotopic nature of the clustering. Secondly, the four non-empty subcells are not filled equally. One cell must get over half the available material and the rest is divided among the other three cells. None of these three cells can be empty, otherwise both the Hausdorff and correlation dimensions would be wrong. It is the least filled of the four cells that makes the dominant contribution to $\tau(-1)$ and so the uncertainty in determining $\tau(-1)$ from the data has little effect on the cells that get most of the material at each level of the hierarchy.

## 5 DISCUSSION

### 5.1 The Lognormal distribution

We have now reached the main objectives of this paper. We have shown that the simple multifractal scaling model that seems to be implied by the observations (Martínez & Jones 1990; Martínez *et al.* 1990) means that the distribution of densities should be roughly Lognormal. There is also some direct observational evidence that, at least on intermediate spatial scales, the distribution of galaxy counts can be approximated by a Lognormal form, although it does seem to break down on very small scales and at very high densities (Hubble 1934; Coles & Jones 1991; Coles & Plionis 1991; Saunders *et al.* 1991).

The probability density (19) should be a reasonable approximation to the distribution of galaxy densities on all scales where the multifractal scaling laws hold. It cannot of course be exactly correct because the $f(\alpha)$ curve is not exactly of the parabolic form we have assumed (17). In particular, the form (17) must be wrong for large $\alpha$ where it would be negative – this corresponds to regions of low density gradient. Nevertheless, the approximation is a good one and has been used in many other contexts (Paladin & Vulpiani 1987). The multiplicative model for $D_q$ provides a slightly better fit to the data (Fig. 3), but does not lead to a simple form for the counts-in-cells distribution.

### 5.2 Analogies with turbulence

It is interesting to consider our result (19) for the counts-in-cells distribution in relation to the simple model for the turbulent dissipation of energy suggested by Kolmogorov

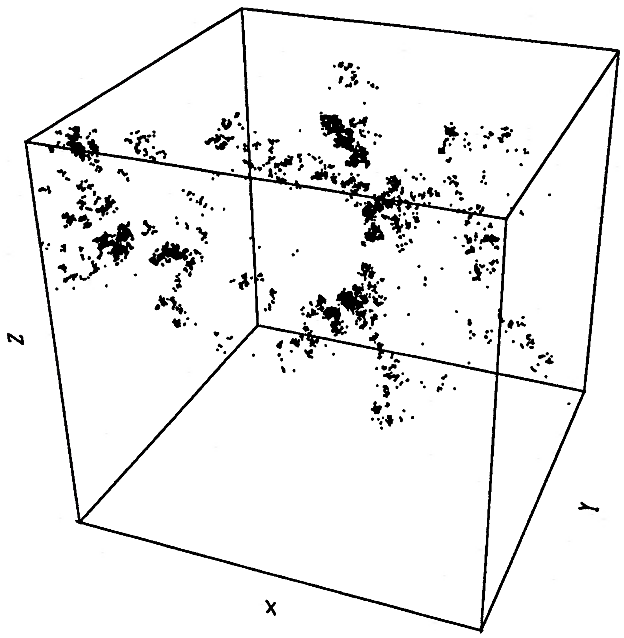


**Figure 2.** The multiplicative random model with four $f_i = 0$ and the others taking the values $f_1 = f_2 = 0.07$, $f_3 = 0.32$ and $f_4 = 0.54$. The model has been evolved until a lattice of $2^7 \times 2^7 \times 2^7$ has been reached. Nearly 8000 points have been distributed according to the probabilities attached to each pixel.

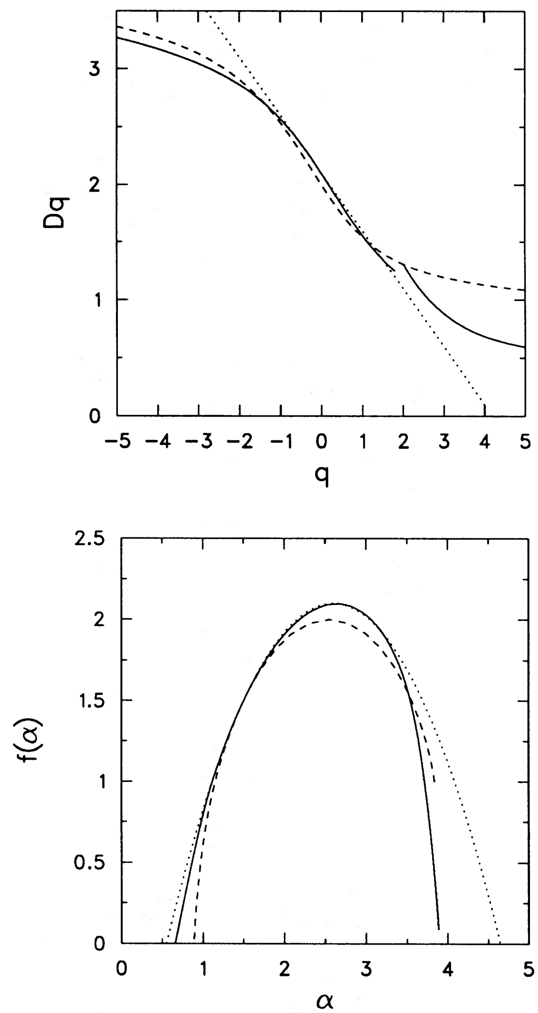


**Figure 3.** The generalized dimensions, $D_q$, and the multifractal spectrum, $f(\alpha)$, for the galaxy distribution of the CfA catalogue (solid line). The parabolic approximation to $f(\alpha)$ yields the $D_q$ plotted as the dotted straight line. The multiplicative multifractal model with appropriate parameters $f_i$ is shown as the dashed line.

(1962). In Kolmogorov's model, the initial assumption is that the dissipation actually followed a Lognormal distribution; multifractal scaling laws are then derived from this. We have merely proceeded in the reverse direction from the empirical scaling laws to the distribution of densities. There are, however, important differences between our treatment of galaxy clustering and the analysis of turbulence. In the theory of turbulence there is a dynamical principle that allows us to fix some points on the $f(\alpha)$ curve. The crucial assumption is that the inertial transfer of energy from large scales to the small dissipative scales is independent of scale in a range of scales, called the inertial range, where such scaling is observed (Kolmogorov 1962). In the special case discussed by Mandelbrot (1974), where the dissipation of the turbulence was assumed to take place on a homogeneous fractal (the $\beta$-model), this is enough to specify completely the geometry of the set of points on which energy is dissipated. Castagnoli & Provenzale (1990) present an attempt to adapt the $\beta$-model to gravitational clustering.

For gravitational clustering there is presently no dynamical analogue of the turbulent energy cascade. We can reasonably expect that the scale-free nature of the gravitational force, which is never repulsive, will lead on sufficiently long time-scales to scaling hierarchies of some kind, and our simple multiplicative model may provide a clue about what is going on. At present we can only fix the $f(\alpha)$ curve empirically.

### 5.3 Self-gravitating hierarchies

There have been some attempts to relate some aspects of the clustering hierarchy to physical processes. Peebles (1974) and Gott & Rees (1975) produced a model for the clustering hierarchy on scales small enough for there to be sufficient time for each stage of the hierarchy to virialize. This gives rise to an estimate of what we call in this paper the correlation dimension, $D_2$, in terms of the power spectral index, $n$, of the initial fluctuation spectrum. An important assumption is that gravitation is the only force responsible for the generation of the hierarchy (i.e. there is no dissipation). This is essentially the view exploited by Efstathiou *et al.* (1979). An alternative picture, where galaxies form by a self-similar fragmentation of gas clouds swept up by expanding void regions, is discussed by Newman & Wasserman (1990).

In both of these situations we expect the Lognormal distribution of densities we have derived in this paper to be a reasonable approximation, both by virtue of the multifractal scaling arguments we have given here and by virtue of general considerations based on the Central Limit Theorem (Coles & Jones 1991). Indeed, Lognormal distributions are ubiquitous in self-similar coagulation and fragmentation models where there is only limited correlation between successive stages of the hierarchy; under these circumstances the resulting distribution resembles that of a product of a large number of uncorrelated splittings or mergings. Just as the distribution of the sum of independent variates tends to a Gaussian by virtue of the Central Limit Theorem, so does the distribution of the logarithm of the product of such variates (Zinnecker 1984; Paladin & Vulpiani 1987; Coles & Jones 1991).

### 5.4 Mechanisms

Such arguments do not provide a complete understanding of the origin of the observed scaling properties. In turbulence, the intermittency arises from local instabilities and bifurcations which can be investigated by writing down the dynamical equations governing the fluid flow. We do not know all the physics required to describe the clustering process in the Universe, so we cannot at this stage say what it is that gives rise to the multifractal behaviour we observe. We have a clue in the fact that the evolution of initially Gaussian fluctuations leads to a non-Gaussian distribution; the importance of voids in such distributions was demonstrated by Coles & Jones (1991), again using a Lognormal distribution. We also have clues from our fragmentation hierarchy, though we have no idea why the parameters $f_i$ $(i=1\ldots4)$ have the values we have found.

We also stress that the description of a point process in terms of its singularity spectrum, $f(\alpha)$, is not complete. We have been able to derive some simple statistical properties of the distribution sampled at a single point by virtue of the local scaling properties of the model. But many different geometric prescriptions possess the same scaling behaviour; see Barnsley (1988) for many examples. A complete description based on the multifractal approach must take account of the spatial geometry of the fractal structures over which the scaling parameter $\alpha$ varies. This question has received some attention recently.

In a recent paper, Saar & Saar (1992) attempt to identify from the data the individual fractal structures that make up the overall multifractal describing the distribution of galaxies. This analysis is based on the mathematical treatment given by Barnsley (1988, section 9.6). It has also been claimed that the wavelet transform makes the scaling properties of multifractals manifest in the sense that a multifractal description is a global description, whereas the wavelet description provides information about structural details (Arneodo, Grasseau & Holschneider 1988; Argoul *et al.* 1989; Martínez, Paredes & Saar 1992). There is evidently plenty of scope for research in such directions.

## 6 CONCLUSIONS

We have modelled the galaxy distribution as a set of points distributed in the regions between voids where there are no galaxies. If the Universe is divided into cells of size $r$ then the fraction of occupied cells is supposed to vary like $r^{3-D_0}$, where $D_0$ is the Kolmogorov capacity of the galaxy distribution. The galaxies are further supposed to be distributed on a set of interwoven fractals called a multifractal. The evidence that a multifractal description is appropriate for real data such as the Z-Cat catalogue was presented by Martínez *et al.* (1990).

On the basis of the assumption that the scaling properties of the galaxy distribution are described by a multifractal, we have shown that the density should be approximately Lognormally distributed and that void regions arise as a consequence of a kind of spatial intermittency, or *heterotopicity*, that is inherent in highly skewed distributions such as the Lognormal. This has far-reaching implications for the nature of the clustering hierarchy that develops in the non-linear clustering regime. We have presented a model of this hierarchy based on a simple multiplicative random process; the model reproduces the scaling properties of non-linear cosmological clustering.

This analysis does not provide us with a physical understanding of the dynamics responsible for the generation of the distribution of scaling indices that occurs when the matter flow is non-linear and galaxies form. In this respect, multifractal analysis is still deficient compared to traditional approaches based on correlation functions – there is no time-scale information in the approach. However, the scaling structure that is revealed in the nature of the hierarchy is of considerable interest, revealing a structure far richer than one might have imagined on the basis of the simple clustering hierarchies or coagulation models. There is considerable scope here for numerical modelling of the non-linear regime. Although we are far from a complete understanding of the scaling properties of galaxy clustering and their physical meaning, we feel that the formalism described here will provide additional insights into the nature and origin of galaxy clustering beyond those obtained from the analysis of correlation functions.

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